# Midterm 1

| Question: | MC Section | 2 | 3 | 4 | 5 | 6 | 7 | 8 | Total |
|---|---|---|---|---|---|---|---|---|---|
| Points: | 35 | 10 | 10 | 10 | 15 | 15 | 15 | 20 | 130 |
| Score: | | | | | | | | | |

No books, no notes, no calculators. Please show all your work, except for the MC Section. Simplify all answers. And please, read the problems carefully. Good luck!

Misc:

- $\mathbb{R}$ denotes the field of real numbers.
- $\mathbb{R}^n$ denotes the $n$-dimensional vector space.

1. **MC Section** This is the only part where you don't need to show work.

   (a) (3 points) When solving three homogeneous equations in four unknowns, what happens?

   √ **There are infinitely many solutions, including the zero solution.**

   ○ The zero solution is the only solution.

   ○ There are no solutions.

   ○ It's impossible to tell unless one knows the specific form of the system.

   (b) (3 points) Let $A$ be an invertible matrix. Which of the following statements, if any, are false?

   ○ The linear transformation associated to $A$ must be both one-to-one and onto.

   √ **Every row of $A$ must contain a leading $1$.**

   ○ The reduced row echelon form of $A$ is the identity matrix.

   (c) (3 points) Which of the following operations on an augmented matrix could change the solution set of a system?

   ○ Adding one row to another row.

   ○ Adding a multiple of one row to another row.

   √ **Multiplying one row by any constant.**

   ○ Interchanging two rows.

   ○ None of the above.

   (d) (3 points) Which is *not* a possibility for the number of solutions for a system of linear equations?

   ○ 0 ○ 1 √ 2 ○ infinite ○ all of the above are possible

(e) (3 points) Find the rank of

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

○ 0 √ 1 ○ 2 ○ 3 ○ 4

(f) (3 points) Find the rank of

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

√ 0 ○ 1 ○ 2 ○ 3 ○ not defined

(g) (3 points) Let $E_1$ and $E_2$ be two elementary matrices corresponding to row operations $o_1$ and $o_2$, respectively. What is the ~~elementary~~ matrix corresponding to row operation "first $o_1$, then $o_2$"?

○ $E_1 E_2$ √ $E_2 E_1$ ○ $E_1 + E_2$ ○ $E_2 + E_1$ ○ none of the above

(h) (2 points) (**False**) Let $E_1$ and $E_2$ be two elementary matrices. Then $(E_1 - E_2)$ is an elementary matrix.

(i) (2 points) (**True**) *(Assume $A$ and $B$ are such that the product $AB$ is defined.)* If $AB = C$ and $C$ has 4 columns, then $B$ has 4 columns.

(j) (2 points) (**True**) A product of elementary matrices is invertible.

(k) (2 points) (**True**) An $n \times n$ matrix with rank $n$ is invertible.

(l) (2 points) (**True**) A homogeneous system of linear equations in $n$ variables with coefficient matrix $C$, whose rank is $n - 1$, has infinitely many solutions.

(m) (2 points) (**False**) If $\boldsymbol{b}$ is one of the columns of $A$, then $\boldsymbol{x} = \boldsymbol{b}$ is a solution to $A\boldsymbol{x} = \mathbf{0}$.

(n) (2 points) (**False**) Let $A$ and $B$ be invertible $n \times n$ matrices. Then $AB = BA$.

2. (10 points) Solve (find all solutions to) the system of linear equations in $a, b, c, d$:

$$\begin{cases} 2a + 2c + 6d = 4(1 + b) \\ a = 2b + c + 3d \\ 3a + 6c + 18d = 9 + 6b \end{cases}$$

**Solution:** Let $\sim$ denote row equivalence. Then, starting at the augmented matrix of the system,

$$\left[\begin{array}{cccc|c} 2 & -4 & 2 & 6 & 4 \\ 1 & -2 & -1 & -3 & 0 \\ 3 & -6 & 6 & 18 & 9 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & -2 & 1 & 3 & 2 \\ 1 & -2 & -1 & -3 & 0 \\ 1 & -2 & 2 & 6 & 3 \end{array}\right]$$
$$\sim \left[\begin{array}{cccc|c} 1 & -2 & 1 & 3 & 2 \\ 0 & 0 & -2 & -6 & -2 \\ 0 & 0 & 1 & 3 & 1 \end{array}\right]$$
$$\sim \left[\begin{array}{cccc|c} 1 & -2 & 1 & 3 & 2 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 3 & 1 \end{array}\right]$$
$$\sim \left[\begin{array}{cccc|c} 1 & -2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

We conclude that

$$\begin{cases} a = 1 + 2u \\ b = u \\ c = 1 - 3v \\ d = v \end{cases}$$

for any $u, v \in \mathbb{R}$.

3. For each of the following row operations, write down an ~~elementary~~ matrix $E$ such that for any $3 \times 3$ matrix $M$, $EM$ is obtained from $M$ by performing that row operation:

(a) (3 points) multiply the first and second row by 1/2;

**Solution:**

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E.$$

(b) (3 points) interchange the first and third rows;

**Solution:**

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = E.$$

(c) (4 points) subtract two times the first row from the second row.

**Solution:**

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E.$$

4. Define a function $t : \mathbb{R}^3 \to \mathbb{R}^3$ by the formula

$$t(u, v, w) = (u - v, 2u + w, 2v + w).$$

(a) (4 points) Write down the matrix $T$ associated with the linear transformation $t$. What is its reduced row echelon form?

**Solution:**

$$\begin{aligned} T = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} &\sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

(b) (6 points) Find two vectors in $\mathbb{R}^3$ whose image under $t$ is the same to show that $t$ is not one-to-one.

**Solution:** Since $T$ has rank 2, there are infinitely many solutions to the system $T\boldsymbol{x} = \mathbf{0}$. Specifically,

$$\begin{bmatrix} 1 & -1 & 0 \\ 2 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} = \mathbf{0},$$

i.e., $(1, 1, -2)$ and $(-1, -1, 2)$ have the same image under $t$.

5. Suppose that the augmented matrix of a system of linear equations in $x_1, x_2, x_3, x_4, x_5$ is

$$A = \left[\begin{array}{ccccc|c} 0 & 0 & 1 & -1 & -1 & -1 \\ 1 & -2 & 0 & 0 & 1 & 3 \\ 0 & 0 & 3 & -3 & -3 & 1 \\ 0 & 0 & 0 & 1 & 2 & 3 \end{array}\right],$$

and is row equivalent to

$$B = \left[\begin{array}{ccccc|c} 1 & -2 & 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{array}\right].$$

(a) (2 points) Which variables are the basic variables?

**Solution:** $x_1$, $x_3$, and $x_4$.

(b) (2 points) Which variables are free variables?

**Solution:** $x_2$ and $x_5$.

(c) (5 points) What are the pivot columns of $A$? *(Do not just say, e.g., "the fifth column". You must write out the column vectors explicitly to get credit.)*

**Solution:** First, third, fourth, and sixth: $(0, 1, 0, 0)$, $(1, 0, 3, 0)$, $(-1, 0, -3, 1)$, and $(-1, 3, 1, 3)$.

(d) (2 points) What is the rank of A?

**Solution:** 4. The rank of coefficient matrix is 3.

(e) (4 points) If the system is consistent, write the solution set in parametric form. Otherwise, explain how you know that the system is inconsistent.

**Solution:** It is not, since the last row in $B$ corresponds to the equation $0 = 4$ which is trivially false.

6. Let

$$A = \begin{bmatrix} \frac{1}{3} & 1 \\ \frac{2}{3} & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 0 \\ -1 & -1 \end{bmatrix}.$$

(a) (2 points) Find $(3A + 4B)$.

**Solution:**

$$3A + 4B = \begin{bmatrix} 1 & 3 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 12 & 0 \\ -4 & -4 \end{bmatrix} = \begin{bmatrix} 13 & 3 \\ -2 & -1 \end{bmatrix}.$$

(b) (4 points) Find $(AB)$.

**Solution:**

$$AB = \begin{bmatrix} (3/3-1) & (0/3-1) \\ (6/3-1) & (0/3-1) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}.$$

(c) (4 points) Find $(B^\top A^\top)$.

**Solution:**

$$(B^\top A^\top) = (AB)^\top = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}.$$

(d) (5 points) Find $(AB)^{-1}$.

**Solution:**

$$\left[\begin{array}{cc|cc} 0 & -1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & 0 \end{array}\right],$$

so

$$(AB)^{-1} = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}.$$

7. Let

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & 2 & 1 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}.$$

(a) (5 points) Find $A^{-1}$.

**Solution:**

$$\left[\begin{array}{ccc|ccc} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 0 & 1 \\ 0 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 5 & 0 & 2 & 1 \\ 0 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & 2 & 1 \\ 0 & 1 & 0 & 2 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right]$$

so

$$A^{-1} = \begin{bmatrix} -5 & 2 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

(b) (2 points) Is $\boldsymbol{b}$ expressible as a linear combination of the columns $\boldsymbol{c_1}, \boldsymbol{c_2}, \boldsymbol{c_3}$ of $A$? Circle one: (**Yes**)

(c) (8 points) If it is, give coefficients $d_1, d_2, d_3$ so that $d_1\boldsymbol{c_1} + d_2\boldsymbol{c_2} + d_3\boldsymbol{c_3} = \boldsymbol{b}$. If it is not, explain why not.

**Solution:** Yes, since $A$ is full rank:

$$A\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \boldsymbol{b} \quad \Leftrightarrow \quad \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = A^{-1}\boldsymbol{b} = \begin{bmatrix} -10 \\ 4 \\ 3 \end{bmatrix}.$$

8. Let $I$ denote the $2 \times 2$ identity matrix, and let

$$A = \begin{bmatrix} -1 & 3 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ -2 & 4 \end{bmatrix}.$$

(a) (10 points) Suppose $X$ satisfies $AX = X - I$. Find $X$.

**Solution:**

$$\begin{aligned} AX &= IX - I \\ AX - IX &= -I \\ (A-I)X &= -I \\ X &= -(A-I)^{-1}, \end{aligned}$$

assuming $(A - I)$ is invertible. Now

$$\left[\ (A-I) \mid I\ \right] = \left[\begin{array}{cc|cc} -2 & 3 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} -2 & 0 & 1 & -3 \\ 0 & 1 & 0 & 1 \end{array}\right]$$
$$\sim \left[\begin{array}{cc|cc} 1 & 0 & -1/2 & -3/2 \\ 0 & 1 & 0 & 1 \end{array}\right] = \left[\ I \mid (A-I)^{-1}\ \right],$$

so

$$X = \begin{bmatrix} 1/2 & 3/2 \\ 0 & -1 \end{bmatrix}.$$

(b) (10 points) Suppose $Y$ satisfies $AYA^{-1} = B$. Find $Y$.

**Solution:** Multiplying both sides on the left by $A^{-1}$, and on the right by $A$, one gets $Y = A^{-1}BA$. Now

$$\left[\ A \mid I\ \right] = \left[\begin{array}{cc|cc} -1 & 3 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & -1 & 3/2 \\ 0 & 1 & 0 & 1/2 \end{array}\right] = \left[\ I \mid A^{-1}\ \right],$$

so

$$Y = \begin{bmatrix} -1 & 3/2 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 0 & 2 \end{bmatrix}. = \begin{bmatrix} -1 & 3/2 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} -2 & 6 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 5 & -3 \\ 1 & 1 \end{bmatrix}.$$